PMM U.S.S.R.,Vol.52,No.5,pp.616-624,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

EQUILIBRIUM OF A HOMOGENEOUS ELASTIC MEDIUM BOUNDED BY A RECTILINEAR STIFF ROD*

S.K. KANAUN

A uniform elastic medium with an inclusion in the shape of an elongated solid of revolution is considered. It is assumed that the elastic moduli of the medium are much small than the elastic moduli of the inclusion (stiff rod). The principal term is constructed for the expansion of the elastic fields in a medium with a stiff rod in a series of small parameters of the problem: the ratio between the characteristic linear dimensions of the inclusion and the ratio between the elastic moduli of the medium and the inclusion. The part of the principal term of the "inner" expansion, the stress field within the rod, that varies slowly along the rod axis, is determined by a method described in /1/. Rods with a different change in the radius of the transverse section along the axis, in the shape of a cylinder, an elongated ellipsoid, and a tapered spindle are considered. By using a well-known integral operator the principal term of the desired expansion of the elastic fields outside the rods is restored according to the known inner expansion.

1. Formulation of the problem. In a uniform elastic medium with elastic modulus tensor c_0 let there be an inclusion with the moduli $c = c_0 + c_1$. The inclusion occupies a domain V which has the shape of a body of revolution with axis Γ and radius $a(z)$, where z is a point on Γ and $a(z)$ is a continuous, piecewise-smooth function. We assume that the length of the inclusion along the axis Γ $(2l)$ considerably exceeds its characteristic radius a, while the stiffness of the inclusion is significantly greater than the stiffness of the medium $(c_0 \cdot c^{-1} = O(\delta),\ \delta \ll 1$ where the dot denotes convolution of the tensors in the two indices).

The stress field $\sigma(x)$ in the medium with the inclusion satisfies the well-known relationship /2/ $(x(x_1, x_2, x_3)$ is a point of the medium)

$$c(x)^{-1} \cdot \sigma(x) + \int_V K(x - x') \cdot c_1 \cdot c^{-1} \cdot \sigma(x')\, dx' = c_0^{-1} \cdot \sigma_0(x) \tag{1.1}$$

$$c(x) = c, \quad x \in V; \quad c(x) = c_0, \quad x \notin V$$

where $\sigma_0(x)$ is the external field applied to the medium whose characteristic scale of variation will be considered to be comparable with the rod length but considerably greater than the characteristic radius a. The kernel $K(x)$ of the integral operator $\mathbf{K}$ in this relationship is expressed in terms of the second derivatives of Green's function G for the medium c_0

$$K_{\alpha\beta\lambda\mu}(x) = -(\nabla_\alpha \nabla_\lambda G_{\beta\mu}(x))_{(\alpha\beta)(\lambda\mu)}, \quad \nabla_\alpha c_0^{\alpha\beta\lambda\mu} \nabla_\lambda G_{\mu\nu}(x) = -\delta(x)\, \delta_\nu^\beta \tag{1.2}$$

Here $\delta(x)$ is the delta function and δ_ν^β is the Kronecker delta.

It follows from (1.1) that the elastic field outside the domain V is restored by means of the values of $\sigma(x)$ within V. The equation for the field $\sigma(x)$ within the rod is obtained by multiplying both sides of (1.1) by the characteristic function of the domain V.

It will later be convenient to consider the tensor components c and c_0 in (1.1) as dimensionless quantities, where $c_0 = O(1),\ c^{-1} = O(\delta)$. For this it is sufficient to multiply both sides of (1.1) by the characteristic value of the elastic modulus of the medium.

In this paper we construct the principal term of the expansion of the field $\sigma(x)$ within the rod in a series in small parameters of the problem $\varepsilon = a/l$ and δ. As follows from the results of /1/, the principal term of the expansion mentioned for $\sigma(x)$ consists of a slowly varying component along the rod axis and functions of boundary-layer type localized in the neighbourhood of singular points of Γ, breaks in the function $a(z)$ or the ends of the rod. The main purpose of the paper is to construct the slowly varying part of the principal term of the expansion of $\sigma(x)$ in series in ε and δ.

*Prikl.Matem.Mekhan.,52,5,789-800,1988

Let us note an important property of the limit of the solution of (1.1) in ε. If we pass to the limit as $\varepsilon \to 0$ in this equation, its solution $\sigma(x)$ turns out to be constant in each section of the rod (see /1/). Consequently, it can be expected that the principal term of the desired expansion of the field $\sigma(x)$ will be constant in transverse sections of the domain V, at least far from the rod ends and points of the break in the function $a(z)$. This allows some simplification of the initial equation (1.1).

We place the origin of a Cartesian system of coordinates y_1, y_2, z at the middle of the rod while directing the z axis along Γ. We let $\omega(z)$ denote the transverse section of the rod. For each point $x \in V$ a unique representation $x = y + zm$ hold, where $y(y_1, y_2)$ is a vector in the plane $\omega(z)$ and m is the direction of the z axis. Assuming $\sigma(x) = \sigma(z)$, we examine (1.1) at points on the rod axis Γ. We introduce the relative coordinates $\xi = z/l$, $\eta = y/l$ and after integration over the transverse sections $\omega(\xi)$ we obtain

$$c^{-1}\cdot\sigma(\xi) + \int_{-1}^{1} \bar{K}(\xi, \xi')\cdot c_1\cdot c^{-1}\cdot\sigma(\xi')\, d\xi' = c_0^{-1}\cdot\sigma_0(\xi) \tag{1.3}$$

$$\bar{K}(\xi, \xi') = \int_{\omega(\xi')} K[(\xi - \xi')m - \eta']\, d\eta' \tag{1.4}$$

It turns out that the components varying slowly along Γ for the principal term of the desired expansion of the function $\sigma(x)$ and the solution of (1.3) are in agreement. Eq.(1.3) also enables us to find the most "far-acting" of the functions of boundary layer type that occur in the expression for the principal term of the expansion $\sigma(x)$.

We will now construct the formal expression for the principal term of the expansion of the solution of (1.3) in a series in the small parameters ε and δ (Sect.3). We will first find the explicit form of the kernel $\bar{K}(\xi, \xi')$ of the operator $\bar{K}$ in (1.3) in Sect.2.

2. Representation of the operator K.

We will introduce the tensor basis $P_i(m)$ that is convenient for representing the quadrivalent tensors occurring in the problem and which we construct from the unit vector m_α and the bivalent tensor $\theta_{\alpha\beta} = \delta_{\alpha\beta} - m_\alpha m_\beta$

$$P_{1\alpha\beta\lambda\mu} = (\theta_{\alpha\mu}\theta_{\beta\mu})_{(\alpha\beta)(\lambda\mu)}, \quad P_{2\alpha\beta\lambda\mu} = \theta_{\alpha\beta}\theta_{\lambda\mu}, \quad P_{3\alpha\beta\lambda\mu} = \theta_{\alpha\beta} m_\lambda m_\mu$$
$$P_{4\alpha\beta\lambda\mu} = m_\alpha m_\beta \theta_{\lambda\mu}, \; P_{5\alpha\beta\lambda\mu} = (\theta_{\alpha\lambda} m_\beta m_\mu)_{(\alpha\beta)(\lambda\mu)}, \; P_{6\alpha\beta\lambda\mu} = m_\alpha m_\beta m_\lambda m_\mu$$

It can be shown that these six linearly independent tensors form a closed algebra relative to the multiplication-convolution operation in two indices. An analgous tensor basis was examined in /3/ (Appendix 4) where a "multiplication" table of the tensors $P_i(m)$ and an inversion formula for the tensor belonging to a linear shell $P_i(m)$ are presented.

Furthermore, we will consider the medium to be isotropic and the rod to be transversely-isotropic with isotropy axis directed along the Γ axis. If m is the direction of the isotropy axis, then the tensors c_0 and c in the basis $P_i(m)$ take the form

$$\begin{aligned} c_0 &= 2\mu_0 P_1 + \lambda_0 (P_2 + P_3 + P_4) + 4\mu_0 P_5 + (\lambda_0 + 2\mu_0) P_6 \\ c &= 2\mu P_1 + \lambda P_2 + \tau (P_3 + P_4) + \gamma P_5 + \rho P_6 \end{aligned} \tag{2.1}$$

Here λ_0, μ_0 are the Lame parameters of the medium, and the relationship between the constants $\lambda, \mu, \tau, \gamma, \rho$ and the technical elastic moduli of a transversely-isotropic body is given by the equalities

$$\begin{aligned} &E_1 = E_2 = \left(\frac{\rho}{2\Delta} + \frac{1}{4\mu}\right)^{-1}, \quad E_m = \frac{\Delta}{2(\lambda + \mu)}, \quad \mu_1 = \mu_2 = \mu \\ &\nu_1 = E_1\left(\frac{1}{4\mu} - \frac{\rho}{2\Delta}\right), \quad \nu_2 = E_2\frac{\tau}{\Delta}, \quad \mu_m = \frac{1}{4}\gamma \\ &\Delta = 2[(\lambda + \mu)\rho - \tau^2] \end{aligned} \tag{2.2}$$

where E_1, E_2, E_m are Young's moduli (E_m is the modulus along the isotropy axis) ν_1, ν_2 and Poisson's ratios, and μ_1, μ_2, μ_m are shear moduli.

The expression for the symbol $K^*(k)$ of the operator K in (1.1) - the Fourier transformation of the function $K(x)$ (1.2) - in the basis $P_i(m)$ has the form

$$K^*(k) = \frac{1}{\mu_0}[P_5(n) + (1 - \varkappa_0) P_6(n)], \quad n = \frac{k}{|k|}, \quad \varkappa_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0} \tag{2.3}$$

where $k(k_1, k_2, k_3)$ is the vector parameter of the Fourier transformation (the k_1, k_2, k_3 coordinate system is conjugate to η_1, η_2, z).

We now turn to (1.3) and we find an expression for the function $\bar{K}(\xi, \xi')$ the kernel of the integral operator $\bar{K}$ in this equation. The equality

$$\bar{K}(\xi,\xi')=(2\pi)^{-3}\int\limits_{\omega(\xi')}d\eta'\int K^*(k)\exp\{-ik[(\xi-\xi')m-\eta']\}\,dk$$

follows from relationships (1.4) and (2.3), where the integration over $k(k_1, k_2, k_3)$ is performed over the whole k-space. Changing the order of evaluating the integrals, first with respect to $\omega(\xi')$, and then with respect to k_1, k_2, we obtain

$$\bar{K}(\xi,\xi')=(2\pi)^{-1}\int\limits_{-\infty}^{\infty}\bar{K}^*(\varepsilon(\xi'),k_3)\,e^{-ik_3(\xi-\xi')}\,dk_3,\quad \varepsilon(\xi)=\frac{a(\xi)}{l} \tag{2.4}$$

where the function $\bar{K}^*(\varepsilon(\xi'), k_3)$ - the symbol of the operator $\bar{K}$ — has the form

$$\begin{aligned}&\bar{K}^*=k_1^*P_1+k_2^*P_2+k_3^*(P_3+P_4)+k_5^*P_5+k_6^*P_6\\&k_1^*=(16\mu_0)^{-1}[4(2-\varkappa_0)T_1^*+\varkappa_0T_2^*],\\&k_2^*=(32\mu_0)^{-1}\varkappa_0(T_2^*-4T_1^*)\\&k_3^*=-(4\mu_0)^{-1}\varkappa_0T_2^*,\quad k_5^*=(2\mu_0)^{-1}(2-T_1^*-2\varkappa_0T_2^*)\\&k_6^*=(2\mu_0)^{-1}[2(1-\varkappa_0)(1-T_1^*)+\varkappa_0T_2^*]\end{aligned} \tag{2.5}$$

The functions $T_1^*(\varepsilon, k)$ and $T_2^*(\varepsilon, k)$ are determined by the relationships (the subscript 3 is here and henceforth omitted from the argument k_3)

$$T_1^*(\varepsilon,k)=\varepsilon(\xi')|k|K_1(\varepsilon(\xi')|k|),\quad T_2^*(\varepsilon,k)=\varepsilon^2(\xi')k^2K_0(\varepsilon(\xi')|k|) \tag{2.6}$$

where K_0 and K_1 are modified Bessel functions.

It is seen hence and from (2.4) and (2.5) that the kernel $\bar{K}(\xi,\xi')$ of the integral operator $\bar{K}$ is governed by a formula analogous to (2.5) in which $T_1^*(\varepsilon, k)$ and $T_2^*(\varepsilon, k)$ should be replaced by the functions

$$\begin{aligned}&T_1(\xi,\xi')=\frac{\varepsilon^2(\xi')}{2[(\xi-\xi')^2+\varepsilon^2(\xi')]^{3/2}},\\&T_2(\xi,\xi')=-\frac{1}{2}\frac{d^2}{d\xi^2}\left\{\frac{\varepsilon^2(\xi')}{[(\xi-\xi')^2+\varepsilon^2(\xi')]^{1/2}}\right\}\end{aligned} \tag{2.7}$$

They are the Fourier transforms $T_1^*(\varepsilon, k)$, $T_2^*(\varepsilon, k)$ in the variable k and are the kernels of the integral operators T_1 and T_2 whose action on the function $\sigma(\xi)$ is governed by the formula

$$(T_i\sigma)(\xi)=\int\limits_{-1}^{1}T_i(\xi,\xi')\,\sigma(\xi')\,d\xi',\quad i=1,2 \tag{2.8}$$

The function $\varepsilon(\xi)$ in (2.4)-(2.7) can be represented in the form of the product $\varepsilon(\xi)=\varepsilon\alpha(\xi)$, $\varepsilon\ll 1$, $\alpha(\xi)=O(1)$, where $\alpha(\xi)$ is a function of the rod shape.

Let us extract the principal terms of the expansion of the symbol $\bar{K}^*$ (2.5) in a series in the small parameter ε. Expanding the Bessel functions K_0 and K_1 in the mentioned series in (2.6) /4/, we will have

$$\begin{aligned}&\bar{K}^*(\varepsilon,k)=A_0+(\varepsilon^2\ln\varepsilon)\,\alpha^2(\xi')\,k^2A_1+O(\varepsilon^2)\\&A_0=(8\mu_0)^{-1}[2(2-\varkappa_0)P_1-\varkappa_0P_2+4P_5]\\&A_1=(8\mu_0)^{-1}[2(1-\varkappa_0)P_1-\varkappa_0P_2+2\varkappa_0(P_3+P_4)+2(4\varkappa_0-1)P_5-4P_6]\end{aligned} \tag{2.9}$$

We will note the properties of the tensor A_0 that are important for later. We introduce the orthogonal projectors Θ and Π (I is the unit quadrivalent tensor)

$$\begin{aligned}&\Theta=P_1+2P_5,\quad \Pi=P_6,\quad \Theta+\Pi=I\\&\Theta\cdot\Theta=\Theta,\quad \Pi\cdot\Pi=\Pi,\quad \Theta\cdot\Pi=\Pi\cdot\Theta=0\end{aligned} \tag{2.10}$$

The linear space drawn by the tensors $P_i(m)$ is divided into two orthogonal subspaces (Θ and Π) by using these projectors. It can be shown that the tensor A_0 belongs to the subspace Θ and has a non-degenerate inverse A_0^{-1} therein

$$\Theta\cdot A_0=A_0\cdot\Theta=A_0,\quad \Pi\cdot A_0=A_0\cdot\Pi=0,\quad A_0\cdot A_0^{-1}=A_0^{-1}\cdot A_0=\Theta \tag{2.11}$$

3. A formal scheme for constructing the principal terms of the solution of (1.3). We will examine a formal procedure for constructing the principal terms of the expansion of the solution of (1.3) in a series in ε and δ. We will retain the first two components in the symbol $\bar{K}^*(\varepsilon, k)$ of the operator $\bar{\mathbf{K}}$ in expansion (2.9). Then (1.3) takes the form $(D = d/d\xi)$

$$c^{-1}\cdot\sigma(\xi) + A_0\cdot c_1\cdot c^{-1}\cdot\sigma(\xi) - \varepsilon^2 \ln \varepsilon D^2[\alpha^2(\xi)\sigma(\xi)] = c_0^{-1}\cdot\sigma_0(\xi) \quad (3.1)$$

We will seek the solution of this equation in the form

$$\sigma(\xi) = \sigma^\circ(\xi) + \varepsilon^2 \ln \varepsilon\sigma^1(\xi) + \ldots \quad (3.2)$$

We obtain the equation for the principal term of this expansion $\sigma^\circ(\xi)$ by substituting (3.2) into (3.1) and retaining terms of the highest order in ε on the left-hand side.

The following cases are possible depending on the relationship between the small parameters ε and δ.

1°. $\delta^{-1}\varepsilon^2 \ln \varepsilon = o(1)$. In this case the equation for $\sigma^\circ(\xi)$ take the form

$$c^{-1}\cdot\sigma^\circ(\xi) + A_0\cdot c_1\cdot c^{-1}\cdot\sigma^\circ(\xi) = c_0^{-1}\cdot\sigma_0(\xi)$$

Acting with the operators Θ and Π on both sides of this equation and taking account of the properties (2.11) of the tensor A_0, we obtain two relationships

$$A_0\cdot\sigma_\Theta^\circ(\xi) + (\Theta - A_0c_0)\cdot c^{-1}\cdot\sigma^\circ(\xi) = \Theta\cdot c_0^{-1}\cdot\sigma_0(\xi) \quad (3.3)$$

$$\Pi\cdot c^{-1}\cdot\sigma^\circ(\xi) = \Pi\cdot c_0^{-1}\cdot\sigma_0(\xi); \quad \sigma_\Theta^\circ = \Theta\cdot\sigma^\circ, \quad \sigma_\pi^\circ = \Pi\cdot\sigma^\circ \quad (3.4)$$

Since A_0 is a non-degenerate tensor in the subspace Θ with components of the order of unity, the estimates $\sigma_\Theta^\circ(\xi) = O(1)$, $\sigma_\pi^\circ(\xi) = O(\delta^{-1})$ follow from (3.3) and (3.4). (It is assumed here that $\sigma_0(\xi) = O(1)$). Hence, taking into account the equality $\sigma_\pi^{\circ\alpha\beta} = \sigma_m^\circ m^\alpha m^\beta$ we obtain the following estimate for the tensor $\sigma^\circ(\xi)$:

$$\sigma^{\circ\alpha\beta}(\xi) = \sigma_m^\circ(\xi)m^\alpha m^\beta + O(1), \quad \sigma_m^\circ(\xi) = O(\delta^{-1}) \quad (3.5)$$

The expression for the axial component $\sigma_m^\circ(\xi)$ of the tensor $\sigma^\circ(\xi)$ follows from (3.4) and (3.5) and has the form

$$\sigma_m^\circ(\xi) = E_m E_0^{-1}[\sigma_{0m}(\xi) - \nu_0 \operatorname{tr}\sigma_{0\Theta}(\xi)] \quad (3.6)$$

where $\operatorname{tr}\sigma_{0\Theta} = (\sigma_{0\Theta})_{\alpha}^{\alpha}$ is the trace of the tensor $\sigma_{0\Theta}$; E_0, ν_0 are the Young's modulus and Poisson's ratio of the medium, and E_m is Young's modulus of the rod.

2°. $\delta^{-1}\varepsilon^2 \ln \varepsilon = O(1)$ $(c_0\cdot c^{-1} = O(\varepsilon^2 \ln \varepsilon))$. In this case the equation for $\sigma^\circ(\xi)$ in (3.2) agrees with (3.1). Acting with the operators Θ and Π on both sides of (3.1), exactly as in case 1°, it can be shown that the tensor $\sigma^\circ(\xi)$ satisfies the estimate (3.5) while the equation for the axial component $\sigma_m^\circ(\xi)$ of this tensor takes the form

$$D^2[\alpha^2(\xi)\sigma_m^\circ(\xi)] - p^2\sigma_m^\circ(\xi) = \frac{1+\nu_0}{\varepsilon^2 \ln \varepsilon}[\sigma_{0m}(\xi) - \nu_0 \operatorname{tr}\sigma_{0\Theta}(\xi)] \quad (3.7)$$

$$p^2 = -\frac{E_0}{E_m}\frac{1+\nu_0}{\varepsilon^2 \ln \varepsilon}, \quad p = O(1)$$

The function $\sigma^\circ(\xi)$ can be found from this differential equation apart from two arbitrary constants. To determine these constants, certain additional conditions (Sect.4) must be imposed on the function $\sigma_m^\circ(\xi)$.

Let us estimate the closeness of the formal expressions obtained for $\sigma^\circ(\xi)$ to the solution of the integral Eq.(1.3). To do this we will investigate the residual from the right-hand side of (1.3), that occurs on substituting the function $\sigma^\circ(\xi)$ in its left-hand side. If $\sigma^\circ(\xi)$ satisfies the estimate (3.5), then the Π-component of the residual turns out to be most essential. Substituting the function $\sigma^\circ(\xi)$ into the left-hand side of (1.3) and acting with the operator Π on the result, we will have

$$\alpha_m\sigma_m^\circ(\xi) - (T\sigma_m^\circ)(\xi) = f_0(\xi) + R(\sigma_m^\circ, \xi) \quad (3.8)$$

$$\alpha_m = 1 + \frac{E_0}{2E_m}\frac{1-\nu_0}{1-\varkappa_0}, \quad T = T_1 - \frac{\varkappa_0}{2(1-\varkappa_0)}T_2$$

$$f_0(\xi) = \frac{1+\nu_0}{1-\varkappa_0}[\sigma_{0m}(\xi) - \nu_0 \operatorname{tr}\sigma_{0\Theta}(\xi)]$$

where $R(\sigma_m^\circ, \xi)$ is the desired residual, and the operators T_1, T_2 are defined by the relationships (2.7) and (2.8).

A component $\sigma_m^\circ(\xi)$ must be added to $\sigma_m^1(\xi)$ to compensate for the residual so that the function $\sigma_m(\xi) = \sigma_m^\circ(\xi) + \sigma_m^1(\xi)$ would satisfy the equation

$$\alpha_m\sigma_m - T\sigma_m = f_0 \quad (3.9)$$

If the component σ_m^1 can be neglected as compared with σ_m°, the function $\sigma_m^\circ m^{\alpha_m t}$ is the principal term of the expansion of the solution of (1.3) in a series in ε and δ. Otherwise, the principal term of the expansion of σ_m^1 in ε and δ should be added to σ_m°. We note that the Θ-component of the residual from the right-hand side of (1.3) is compensated by components of the order of unity that are small compared with the axial component of the order of δ^{-1} on substituting a function $\sigma^\circ(\xi)$ of the form (3.5) into the left-hand side Eq.(1.3).

We examine the form of the residual R in (3.8) for a rod with a different law of variation of the radius of the transverse section along the ξ axis.

4. Cylindrical rod. For a cylindrical rod $\varepsilon = a/l$, $\alpha(\xi) = 1$. We will estimate the result of the action of the operators T_1 and T_2 in (2.8) on a smooth bounded function $\sigma(\xi)$ of the order of unity. We will represent $(T_1\sigma)(\xi)$ in the form

$$(T_1\sigma)(\xi) = T_1^\circ(\xi)\sigma(\xi) + T_1^1(\xi)D\sigma(\xi) + \frac{1}{2}T_1^2(\xi)D^2\sigma(\xi) + \int_{-1}^{1} T_1(\xi,\xi')\Big[\sigma(\xi') - \sigma(\xi) - D\sigma(\xi)(\xi'-\xi) - \frac{1}{2}D^2\sigma(\xi)(\xi'-\xi)^2\Big]d\xi' \tag{4.1}$$

$$T_1^k(\xi) = \int_{-1}^{1} T_1(\xi,\xi')(\xi'-\xi)^k d\xi', \quad k = 0, 1, 2 \tag{4.2}$$

where the kernel $T_1(\xi,\xi')$ has the form (2.7) for $\varepsilon(\xi) = \varepsilon$.

Evaluating the integrals (4.2) and substituting the result into (4.1), we obtain the estimate

$$(T_1\sigma)(\xi) = \sigma(\xi) - \Big[\Phi_0\Big(\frac{1-\xi}{\varepsilon}\Big) + \Phi_0\Big(\frac{1+\xi}{\varepsilon}\Big)\Big]\sigma(\xi) - \varepsilon\Big[\Phi_1\Big(\frac{1-\xi}{\varepsilon}\Big) - \Phi_1\Big(\frac{1+\xi}{\varepsilon}\Big)\Big]D\sigma(\xi) - \frac{1}{2}\varepsilon^2\ln\varepsilon D^2\sigma(\xi) + \frac{1}{2}\varepsilon^2\ln\varepsilon[\Phi_2(1-\xi,\varepsilon) + \Phi_2(1+\xi,\varepsilon)]D^2\sigma(\xi) + O(\varepsilon^2) \tag{4.3}$$

It is here taken into account that the integral component in (4.1) is of the order of ε^2; Φ_0, Φ_1, Φ_2 are functions of boundary-layer type localized in the neighbourhood of the rod ends $\xi = \pm 1$

$$\Phi_0(t) = \frac{1}{2}\operatorname{sign} t\Big(1 - \frac{|t|}{\sqrt{t^2+1}}\Big), \quad \Phi_1(t) = \frac{1}{2}\frac{1}{\sqrt{t^2+1}} \tag{4.4}$$

$$\Phi_2(t,\varepsilon) = 1 - {}^1/_2(\ln\varepsilon)^{-1}\ln(\sqrt{t^2+\varepsilon^2} - |t|)$$

The estimate

$$(T_2\sigma)(\xi) = -\varepsilon\Big[\Phi_1\Big(\frac{1-\xi}{\varepsilon}\Big) - \Phi_1\Big(\frac{1+\xi}{\varepsilon}\Big)\Big]D\sigma(\xi) - \varepsilon^2\ln\varepsilon D^2\sigma(\xi) - \frac{1}{2}\varepsilon^2\ln\varepsilon[\Phi_2(1-\xi,\varepsilon) + \Phi_2(1+\xi,\varepsilon)]D^2\sigma(\xi) + O(\varepsilon^2) \tag{4.5}$$

can be obtained analogously.

We will examine the case when $\delta^{-1}\varepsilon^2\ln\varepsilon = o(1)$ and the function $\sigma^\circ(\xi)$ has the form (3.6). Substituting (3.6) into (3.8) and taking account of (4.3) and (4.5), for the residual R we obtain the expression

$$R = \Big[\Phi_0\Big(\frac{1-\xi}{\varepsilon}\Big) + \Phi_0\Big(\frac{1+\xi}{\varepsilon}\Big)\Big]\sigma_m^\circ(\xi) + O(\varepsilon\sigma_m^\circ) \tag{4.6}$$

To compensate the principal term of the residual that is of the order of σ_m° in the neighbourhood of the rod ends, components σ_m^+ and σ_m which depend on the "fast" variables $\tau_+ = (1-\xi)/\varepsilon$ and $\tau_- = (1+\xi)/\varepsilon$ must be added to the function σ_m°. The equations for the functions $\sigma_m^\pm$ follow from (3.8) and have the form

$$\alpha_m\sigma_m^\pm(\tau_\pm) - (T_\tau\sigma_m^\pm)(\tau_\pm) = \Phi_0(\tau_\pm)\sigma_m^\circ(\pm 1) \tag{4.7}$$

$$(T_\tau\sigma_m)(\tau) = \int_0^\infty T_\tau(\tau-\tau')\sigma_m(\tau')d\tau'$$

$$T_\tau(\tau) = \frac{1}{2}(\tau^2+1)^{-3/2}\Big[1 - \frac{\varkappa_0}{1-\varkappa_0}\Big(1 - \frac{3\tau^2}{\tau^2+1}\Big)\Big]$$

The numerical solution of the Wiener-Hopf equation analogous to (4.7) was considered in /1/. If $\delta^{-1}\varepsilon^2 \ln \varepsilon = o(1)$ then the functions $\sigma_m^+((1-\xi)/\varepsilon)$ and $\sigma_m^-((1+\xi)/\varepsilon)$, respectively are of the order σ_m° for $\xi = 1$ and $\xi = -1$ and are functions of boundary-layer type localized in the neighbourhood of the rod ends. The remaining part of the residual is compensated by components of the order of $\varepsilon\sigma_m^\circ$ in the expression for σ_m, that can be neglected compared with the principal term $\sigma_m^\circ(\xi) + \sigma_m^+((1-\xi)/\varepsilon) + \sigma_m^-((1+\xi)/\varepsilon)$.

We will now examine the case when $\delta^{-1}\varepsilon^2 \ln \varepsilon = O(1)$. The function $\sigma^\circ(\xi)$ here satisfies (3.7) (for $\alpha(\xi) = 1$) and depends on two arbitrary constants. On substituting the solution (3.7) into the left-hand side of (3.8) and taking account of (4.3) and (4.5) the residual R has the form (4.6) as before. The principal term of the residual is obviously a minimum if the function $\sigma_m^\circ(\xi)$, as a solution of (3.7), also satisfies the conditions

$$\sigma_m^\circ(-1) = \sigma_m^\circ(1) = 0 \tag{4.8}$$

These conditions enable the constants in the general solution of (3.7) to be determined. If $\sigma_m^\circ(\xi)$ satisfies (3.7) and (4.8), then the expression for the residual R in (3.8) takes the form

$$R = \varepsilon\left[\Psi\left(\frac{1+\xi}{\varepsilon}\right) D\sigma_m^\circ(-1) - \Psi\left(\frac{1-\xi}{\varepsilon}\right) D\sigma_m^\circ(1)\right] + O(\varepsilon^2 \ln \varepsilon\sigma_m^\circ) \tag{4.9}$$

$$\Psi(t) = \Phi_3(t) - \frac{\varkappa_0}{2(1-\varkappa_0)}\Phi_1(t), \quad \Phi_3(t) = {}^1/_2\left(\sqrt{t^2+1} - |t|\right)$$

where $\Psi((1 \pm \xi)/\varepsilon)$ is a function of boundary-layer type. To compensate the principal term of the residual that is now of the order of $\varepsilon D\sigma_m^\circ$ in the neighbourhood of the rod ends, the components $\varepsilon g^-((1+\xi)/\varepsilon)$ and $\varepsilon g^+((1-\xi)/\varepsilon)$ should now be added to $\sigma_m^\circ(\xi)$. The equation for the functions g^- and g^+ have a form analogous to (4.7) with right-hand sides $\Psi(\tau_-) D\sigma_m^\circ(-1)$ and $-\Psi(\tau_+) D\sigma_m^\circ(1)$, respectively. The components εg^- and εg^+ can be neglected everywhere as compared with $\sigma_m^\circ(\xi)$ with the exception of ε-neighbourhoods of the rod ends since the function $\sigma_m^\circ(\xi)$ tends to zero as $\xi \to \pm 1$ by virtue of (4.8).

We note that for $p \gg 1(\delta^{-1}\varepsilon^2 \ln \varepsilon = o(1))$ the solution of (3.7) for $\alpha(\xi) = 1$ with conditions (4.8) differs slightly from (3.6). Neighbourhoods of the rod ends that are domains of an exponential boundary layer are the exception. Therefore, to construct the slowly varying component of the principal term of the expansion of the function $\sigma(\xi)$ in ε and δ it is sufficient to examine the case $\delta^{-1}\varepsilon^2 \ln \varepsilon = O(1)$. Other modifications of the relationships between ε and δ can also be investigated for this case by using appropriate passages to the limit in the expression $\sigma^\circ(\xi)$. We note that functions of boundary-layer type that occur in the principal term of the asymptotic form of the solution for $\delta^{-1}\varepsilon^2 \ln \varepsilon = o(1)$ are not exponential but power-law (see (4.7)). Consequently, the function $\sigma^\circ(\xi)$ satisfying (3.7) and (4.8) in the case $p \gg 1$ describes the behaviour of the principal term of the solution (3.9) in the neighbourhood of the rod ends only qualitatively.

The accuracy of the approximation of the solution of (3.9) by the functions $\sigma^\circ(\xi)$ satisfying (3.7) and (4.8) will be examined in the next example. We find the solution of a model equation analogous to (3.9)

$$\alpha_m\sigma - T_1\sigma = -{}^1/_2\varepsilon^2 \ln \varepsilon, \quad \alpha_m = 1 - {}^1/_2 p^2\varepsilon^2 \ln \varepsilon \tag{4.10}$$

where the operator T_1 is defined by the relationships (2.7) and (2.8) for $\varepsilon(\xi) = \varepsilon$.

The procedure described for constructing the principal term of the expansion of the solution (4.10) in a series in ε results in the differential equation $D^2\sigma^\circ - p^2\sigma^\circ = -1$ with homogeneous boundary conditions (4.8). We hence have the expression

$$\sigma^\circ(\xi) = p^{-2}(1 - \operatorname{ch} p\xi/\operatorname{ch} p). \tag{4.11}$$

We will compare the expression obtained for the principal term of the expansion of the solution (4.10) in a series in ε with the results of a numerical solution of this equation as represented in Fig.1 ($\varepsilon = 0.1$) and Fig.2 ($\varepsilon = 0.01$) by continuous lines, and the function $\sigma^\circ(\xi)$ from (4.11) by dashed lines. Values of the parameter $p = 0.4, 1.2, 2, 10$ correspond to curves $1-4$. It is seen that the difference between $\sigma^\circ(\xi)$ and $\sigma(\xi)$ is substantial only in the neighbourhood of the rod ends.

Remark. We examine the residual from the right side of the initial Eq.(1.1) on substituting the function $\sigma^\circ(\xi)$ satisfying (3.7) and (4.8) into its left-hand side. It can be shown that if $\sigma^\circ(\xi)$ is the principal term of the solution of (1.3), then the residual mentioned is represented in the form $R = Q(|\eta| D)\sigma^\circ(\xi)$ where $Q(t)$ is an analytic function whose expansion starts with terms linear in t /1/. The components compensating this part of the residual in the expression for $\sigma(x)$ are of the order of $\varepsilon\sigma^\circ(\xi)$ everywhere with the exception of the neighbourhood of the rod ends a power-law boundary layer domain.

The function $\sigma^\circ(\xi)$ satisfying (3.7) and (4.8) was compared with the exact solution of the problem on the tension of a medium with a stiff cylindrical rod in /5/. This solution was constructed by the finite-elements method. It turned out that the deviation of $\sigma^\circ(\xi)$ from $\sigma(x)$ was considerable only in the neighbourhood of the rod ends, which corresponds to the estimates in this paper.

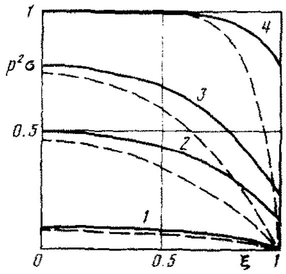

Fig.1

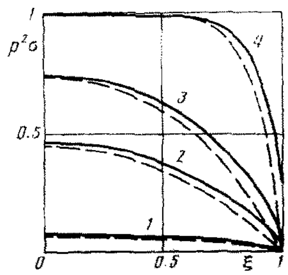

Fig.2

5. An ellipsoidal rod. In this case $\varepsilon(\xi) = \varepsilon\sqrt{1-\xi^2}$, $\alpha(\xi) = \sqrt{1-\xi^2}$. If $\sigma(\xi)$ is a smooth bounded function of order unity, then the estimate for $(T_1\sigma)(\xi)$ and $(T_2\sigma)(\xi)$ that can be obtained in the same way as (4.3) and (4.5) take the form

$$(T_1\sigma)(\xi) = \sigma(\xi) - {}^1/_2\varepsilon^2 \ln \varepsilon D^2 [(1-\xi^2)\sigma(\xi)] + O(\varepsilon^2)$$

$$(T_2\sigma)(\xi) = \varepsilon^2 \ln \varepsilon D [(1-\xi^2)\sigma(\xi)] + O(\varepsilon^2)$$

Therefore, unlike the case of a cylindrical rod, the principal terms of the expansions of $(T_1\sigma)(\xi)$ and $(T_2\sigma)(\xi)$ in series in ε do not contain functions of boundary-layer type. It hence follows that if the function $\sigma_m^\circ(\xi)$ is bounded and satisfies Eq.(3.7) for $\alpha(\xi) = \sqrt{1-\xi^2}$

$$D^2[(1-\xi^2)\sigma_m^\circ(\xi)] - p^2\sigma_m^\circ(\xi) = \frac{1-\varkappa_0}{\varepsilon^2 \ln \varepsilon} f_0(\xi) \tag{5.1}$$

then the residual R in (3.8) takes the form

$$R = O(\varepsilon^2\sigma_m^\circ) \tag{5.2}$$

($f_0(\xi)$ and p^2 are defined in (3.7) and (3.8)).

We note that each of the two linearly independent solutions of the homogeneous Eq.(5.1) has a singularity of the type $(1-\xi)^{-1}$ or $(1+\xi)^{-1}$ in the neighbourhood of the points $\xi = \pm 1$. Consequently, the boundedness condition for $\sigma_m^\circ(\xi)$ is sufficient to determine the constants in the general solution of (5.1). The components in the expression for $\sigma(\xi)$, that compensate the residual (5.2) in (3.8), are of the order of $(\ln\varepsilon)^{-1}\sigma_m^\circ$ for all $\xi \in [-1, 1]$ and can be neglected compared with σ_m°.

If f_0 is a constant function on Γ, then the bounded solution of (5.1) has the form

$$\sigma_m^\circ = (\varepsilon^2 \ln \varepsilon)^{-1} (2 + p^2)^{-1} (\varkappa_0 - 1) f_0 \tag{5.3}$$

and is also constant. We note that in the case of a constant external field, the initial Eq.(1.1) for an ellipsoidal domain V has a known exact solution /2/. It can be shown that the principal term of the asymptotic form of the exact solution as $\varepsilon \to 0$ has the form $\sigma^{\alpha\beta} = \sigma_m^\circ m^\alpha m^\beta$ where σ_m° is given by relationship (5.3).

6. A rod in the shape of a tapered spindle. In this case the function $\varepsilon(\xi)$ has the form $\varepsilon(\xi) = \varepsilon(1 - |\xi|)$, $\alpha(\xi) = 1 - |\xi|$.

We again turn to estimates of the action of the operators T_1 and T_2 on a smooth bounded function $\sigma(\xi)$ of the order of unity. We start with the operator T_1 which we represent in the form

$$T_1 = T_1^- + T_1^+ \tag{6.1}$$

$$(T_1^-\sigma)(\xi) = \int_{-1}^{0} T_1^-(\xi, \xi')\sigma(\xi')\,d\xi', \quad (T_1^+\sigma)(\xi) = \int_0^1 T_1^+(\xi, \xi')\sigma(\xi')\,d\xi'$$

$$T_1^\pm(\xi, \xi') = \frac{1}{2}\frac{\varepsilon^2(1\mp\xi')^2}{[(\xi-\xi')^2 + \varepsilon^2(1\mp\xi')^2]^{3/2}}$$

Here $H_+(\xi)$ is the Heaviside function $(H_+(\xi) = 1,\ \xi > 0;\ H_+(\xi) = 0,\ \xi < 0)$; $H_-(\xi) = H_+(-\xi)$. Let $\sigma_+(\xi)$ be a smooth function given on the positive half-axis $\xi\,(R_+)$ and defined on the negative half-axis R_- by using the analytic continuation procedure. We denote the analogous function given initially on R_- by $\sigma_-(\xi)$.

Representing the function $(T_1^\pm\sigma_\pm)(\xi)$ in a form analogous to (4.1) and evaluating the integrals occurring there, we obtain

$$(T_1\sigma)(\xi) = \sigma(\xi) - \frac{1}{2}\varepsilon^2 \ln \varepsilon D^2 [(1-|\xi|)^2\sigma(\xi)] + \tag{6.2}$$

$$\Phi_0(\xi/\varepsilon)(\sigma_-(\xi)-\sigma_+(\xi))-\varepsilon\Phi_1(\xi/\varepsilon)[\sigma_-(\xi)+\sigma_+(\xi)+$$
$$D\sigma_-(\xi)-D\sigma_+(\xi)]-\frac{1}{2}\varepsilon^2\ln\varepsilon\Phi_2(\xi,\varepsilon)D^2[(1+\xi)^2\sigma_-(\xi)+$$
$$(1-\xi)^2\sigma_+(\xi)]-\frac{1}{2}\varepsilon^2[\sigma_-(\xi)\ln(1+\xi)+\sigma_+(\xi)\ln(1-\xi)]+O(\varepsilon^2)$$
$$\sigma(\xi)=\sigma_-(\xi)H_-(\xi)+\sigma_+(\xi)H_+(\xi)$$

where the functions Φ_0, Φ_1, Φ_2 are defined by relationships (4.4).

If $\sigma_\pm(0)\neq 0$, then the singular component that occurs on differentiating of $|\xi|$ must be discarded here. With this stipulation it is possible to write

$$(T_2\sigma)(\xi)=\varepsilon^2\ln\varepsilon D^2[(1-|\xi|)^2\sigma(\xi)]+ \tag{6.3}$$
$$\frac{1}{2}\varepsilon^2D^2[(1-|\xi|)^2\ln(1-|\xi|)\sigma(\xi)]+O(\varepsilon^2)$$

We now examine (3.7) for the principal term of the expansion of the solution of (1.3) in a series in ε and δ $(\delta^{-1}\varepsilon^2\ln\varepsilon=O(1))$ in the case of a spindle. We rewrite this equation in the form

$$D^2[(1-|\xi|)^2\sigma_m{}^\circ(\xi)]-2q\sigma_m{}^\circ(\xi)=\frac{1-\varkappa_0}{\varepsilon^2\ln\varepsilon}f_0(\xi),\quad 2q=p^2 \tag{6.4}$$

(the quantity p^2 and the function f_0 are defined in (3.7) and (3.8)). Here, as in (6.2) and (6.3), the singular component that occurs in the differentiation of $|\xi|$ should be discarded if $\sigma_m{}^\circ(0)\neq 0$.

The general solution of (6.4) has the form

$$\sigma_m{}^\circ(\xi)=\bar\sigma_m{}^\circ(\xi)+C_+(1-|\xi|)^{\beta_+}+C_-(1-|\xi|)^{\beta_-} \tag{6.5}$$
$$\beta_\pm=-(1/2)(3\pm\sqrt{1+8q})$$

where $\bar\sigma_m{}^\circ(\xi)$ is a bounded particular solution of (6.4), and C_+, C_- are arbitrary constants $(q>0)$.

Since the elastic field in the neighbourhood of a conical singularity on the boundary of a medium and inclusion should be square integrable /6/, the constant C_+ in (6.5) in front of the function $(1-|\xi|)^{\beta_+}$ non-integrable in $[-1, 1]$, will be taken equal to zero.

Substituting (6.5) for $C_+=0$ into the left-hand side of (3.8) and using the estimates (6.2) and (6.3), we obtain that the residual in the neighbourhood of the middle of the rod $(\xi=0)$ is estimated as follows

$$R=2\varepsilon[\Phi_1(\xi/\varepsilon)-(\xi/\varepsilon)\Phi_0(\xi/\varepsilon)]C\beta+$$
$$2\varepsilon\Phi_1(\xi/\varepsilon)[\bar\sigma_m{}^\circ(0)+C]+O(\varepsilon^2\ln\varepsilon\sigma_m{}^\circ),\quad \beta=\beta_-,\ C=C_-$$

It is seen hence and from (4.4) that the principal term of the residual damps out with the asymptotic form $|\xi/\varepsilon|^{-1}$ with distance from the middle of the rod. If the constant C is selected from the condition that the coefficient for this asymptotic form equals zero, then

$$C=-2(2+\beta)^{-1}\sigma_m{}^\circ(0)$$

The asymptotic form of the damping of the principal term of the residual is here determined by the function $|\xi/\varepsilon|^{-3}$ while the expression for $\sigma_m{}^\circ(\xi)$ in (6.7) takes the form

$$\sigma_m{}^\circ(\xi)=\bar\sigma_m{}^\circ(\xi)-2(2+\beta)^{-1}\bar\sigma_m{}^\circ(0)(1-|\xi|)^\beta \tag{6.6}$$

In particular if f_0 in (6.4) is a constant then

$$\sigma_m{}^\circ(\xi)=\frac{(1-\varkappa_0)}{2\varepsilon^2\ln\varepsilon}\frac{f_0}{(1-q)}\left[1-\frac{2}{2+\beta}(1-|\xi|)^\beta\right],\quad q\neq 1 \tag{6.7}$$
$$\sigma_m{}^\circ(\xi)=\frac{\varkappa_0-1}{6\varepsilon^2\ln\varepsilon}f_0[1-2\ln(1-|\xi|)],\quad q=1;\quad q=-\frac{E_0(1+\nu_0)}{2E_m\varepsilon^2\ln\varepsilon}$$

It follows from (6.6), (6.2) and (6.3) that the residual R in (3.8) has a logarithmic singularity $R\sim\varepsilon^2\ln(1-|\xi|)\sigma_m{}^\circ(\xi)$ as a minimum in the neighbourhood of the rod ends. To compensate this part of the residual, functions of boundary-layer type that define the principal term of the solution in the neighbourhood of the rod ends, should be added to $\sigma_m{}^\circ(\xi)$. The equation for these functions can be obtained in the same way as (4.7).

Let us compare (6.6) with the numerical solution of the integral Eq.(3.9). We will make such a comparison using the example of a model equation analogous to (3.9)

$$\alpha_m\sigma-T_1\sigma=-\frac{1}{2}\varepsilon^2\ln\varepsilon,\quad \alpha_m=1-q\varepsilon^2\ln\varepsilon \tag{6.8}$$

where the operator T_1 is defined by the relationships (2.7) and (2.8) for $\varepsilon(\xi)=\varepsilon(1-|\xi|)$. Applying the scheme elucidated above, we find that the expression for the principal term $\sigma^\circ(\xi)$

of the expansion of the equation in a series in ε has the form of (6.7) for $f_0 = (\varkappa_0 - 1)^{-1}\varepsilon^2 \ln \varepsilon$.

The numerical solutions of (6.8) are represented in Fig.3 ($\varepsilon = 0.1$) and Fig.4 ($\varepsilon = 0.01$) by solid lines, and the function $\sigma^\circ(\xi)$ of the form (6.7) by dashed lines. Values of the parameter $q = 0.4, 1.2, 2, 10$ correspond to curves *1-4*. The discrepancy between the solid and dashed curves can be reduced by adding functions of boundary-layer type localized in the neighbourhood of the middle of the rod and its ends to $\sigma^\circ(\xi)$.

7. The principal term of the external expansion of the field $\varepsilon(x)$. In conclusion, we write down the principal term of the external expansion for the strain field $\varepsilon(x)$ in a medium with a stiff rod. In the case of an isotropic medium, the kernel $K(x)$ of the operator **K** in (1.1), exactly like Green's function $G(x)$ with which $K(x)$ is connected by the relationships (1.2), can be written down explicitly. Since for $x \overline{\in} V$ the function $K(x - x')$ in (1.1) is smooth and bounded, the equality

$$\varepsilon(x) = \varepsilon_0(x) - \int_\Gamma K(x - z)\cdot\bar{\sigma}(z)\,dz + O(\varepsilon, \delta) \tag{7.1}$$

$$\bar{\sigma}(z) = \int_{\omega(z)} \sigma(y + zm)\,dy, \quad \varepsilon(x) = c_0^{-1}\cdot\sigma(x), \quad \varepsilon_0(x) = c_0^{-1}\cdot\sigma_0(x)$$

follows from (1.1) at distances from the rod axis substantially exceeding its transverse dimension.

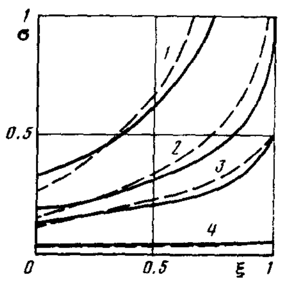

Fig.3

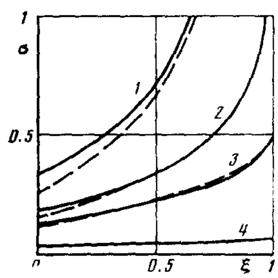

Fig.4

It is taken into account here that $c_1\cdot c^{-1} = I + O(\delta)$, and the coordinates y_1, y_2, z are defined in Sect.2.

Substituting the principal term $\sigma^\circ(z)$ of the expansion of the field $\sigma(x)$ within the rod in a series in ε, δ here we will have

$$\bar{\sigma}^{\alpha\beta}(z) = \sigma_m{}^\circ(z)\,s(z)\,m^\alpha m^\beta + o(\sigma_m{}^\circ s), \quad s(z) = \pi a^2(z) \tag{7.2}$$

We note that functions of boundary-layer type in the expression for $\sigma^\circ(z)$ make a negligibly small contribution (as compared with the component $\sigma^\circ(z)$ varying slowly along Γ) to the magnitude of the elastic stresses and strains far from the rod. To calculate these strains, relationships (7.1) and (7.2) can be used, where the function $\sigma_m{}^\circ(z)$ is determined from (3.7) with the additional conditions dependent on the rod shape (Sects.4-6).

REFERENCES

1. KANAUN S.K., Stationary field in a uniform medium perturbed by an inclusion in the shape of a curvilinear rod, PMM, 51, 2, 1987.
2. KUNIN I.A. and SOSNINA E.G., Ellipsoidal inhomogeneity in an elastic continuous medium, Dokl. Akad. Nauk SSSR, 199, 3, 1971.
3. KUNIN I.A., Theory of Elastic Media with a Microstructure. Nauka, Moscow, 1975.
4. ABRAMOWITZ M. and STEGUN I.M., Handbook on Special Functions with Formulas, Graphs, and Mathematical Tables, Nauka, Moscow, 1979.
5. NIKISHKOV G.P. and CHEREPANOV G.P., Tension of an elastic space with an isolated stiff rod, PMM, 48, 3, 1984.
6. KONDRAT'YEV V.A., Boundary value problems for elliptic equation in domains with conical and angular points, Trudy, Moskov, Matem. Obshch., 16, 1967.

Translated by M.D.F.